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LETTER TO THE EDITOR

Current–voltage characteristics of superconducting arrays

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Abstract. A two-dimensional superconducting array with applied external currents is studied both analytically and numerically. With the Hamiltonian describing the stationary state of the system, we calculate the current-voltage characteristics and, remarkably, obtain Ohmic behaviour in the low-current regime, thus providing a natural explanation of the anomalous behaviour observed in experiments. We also present results of numerical simulations of the Langevin equation ruling the time evolution of the system, which confirm the linear current-voltage relation obtained analytically. Implications for the dynamic KT theory are also discussed.

Two-dimensional (2D) arrays of superconductors, weakly coupled by Josephson junctions, have been extensively studied both experimentally and theoretically (for a list of references see articles in [1]). They provide very interesting model systems for the study of phase transitions and dynamical behaviour from a fundamental point of view, as well as for the understanding of practical superconducting systems such as granular films. In recent years analytical work [2] as well as numerical simulations [3] on 2D arrays of resistively-shunted Josephson junctions have been carried out. In [2] the dynamic response of the system to an alternating field was considered for the case of periodic boundary conditions, and the frequency-dependent conductivity was calculated. However, the response to a (direct) current with realistic boundary conditions and the corresponding current-voltage (I-V) characteristic was not considered. On the other hand, the authors of [3] considered the system in the presence of applied direct currents, which resembles the experimental situation more closely. They calculated the I-V characteristics and time-dependent voltage together with its power spectrum, and demonstrated the existence of a phase transition, which appears to be a Kosterlitz-Thouless (KT) transition [4]. Then there appeared an analytical study of the same system, confirming the KT transition at the current-dependent critical temperature [5]. These results are consistent with recent experiments [6] which measured current-voltage characteristics, and obtained the universal jump in the exponent of the current–voltage relation, which is believed to be characteristic of the KT transition according to the dynamical extension of the KT theory [7]. However, it is of interest to note that in these experiments the I-V relation apparently tends to be linear in the low-current regime, even at temperatures lower than the estimated critical temperature. This *anomalous* behaviour, usually attributed to ambient magnetic fields, has not received serious attention, and a convincing explanation as to its origin still seems to be lacking.

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The purpose of this work is to investigate in detail the $I-V$ characteristics found in numerical simulations and experiments. We consider a 2D array of resistively-shunted Josephson junctions whose time evolution is governed by coupled Langevin equations. With the effective Hamiltonian describing the stationary state of the system, we calculate the $I-V$ characteristics to obtain linear behaviour in the low-current limit. We also present results of numerical simulations, performed with the Langevin equations. They indeed lead to the expected nonlinear behaviour in the appropriate (intermediate-current) regime, in agreement with the standard theory. In addition, in the low-current regime they do display a remarkably linear behaviour, providing a natural explanation of the anomalous behaviour observed in experiments.

We consider a square $N \times N$ array with a resistively-shunted Josephson junction on each bond. We follow [3], and choose the boundary conditions. Along one edge of the array ($x=0$) a uniform current I is injected into each node, while along the opposite edge ($x=N$) the same current I is extracted from each node. With these boundary conditions, the time evolution of the system is governed by the coupled Langevin equations [5]

$$\frac{\mathrm{d}\phi_i}{\mathrm{d}t} = h_i - \left(\frac{2ek_\mathrm{B}T}{\hbar I_\mathrm{c}}\right)^{1/2} \sum_j \sum_k{}' G_{ij}\eta_{jk} \tag{1}$$

with

$$h_i \equiv \sum_j G_{ij} I_j - \sum_j \sum_k{}' G_{ij} \sin(\phi_j - \phi_k)$$

where ϕ_i is the phase of the superconducting order parameter at site $i \equiv (i_x, i_y)$, I_c is the critical current of the junction, and $I_j \equiv I(\delta_{i_x,0} - \delta_{i_x,N})$ is the current injected into node j. In equation (1) the prime restricts the summation to be performed over the nearest neighbours of j, time t has been rescaled in units of $\hbar/2eRI_\mathrm{c}$ with R being the shunt resistance, G_{ij} is the lattice Green function defined by $\Sigma_j' (\phi_i - \phi_j) = \Sigma_j G_{ij}^{-1}\phi_j$, and the (dimensionless) noise current η_{ij} is characterized by $\langle \eta_{ij}(t+\tau)\eta_{kl}(t)\rangle = 2\delta(\tau)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$.

From the Langevin equations (1), one can derive the corresponding Fokker–Planck equation, the stationary solution of which has the form of a Gibbs measure for the canonical distribution with effective *Hamiltonian* [5]

$$H = -J_0 \sum_{\langle ij\rangle} \cos(\phi_i - \phi_j) - \sum_i J_i \phi_i \tag{2}$$

where $J_0 \equiv \hbar I_\mathrm{c}/2e$, and $J_i \equiv J(\delta_{i_x,0} - \delta_{i_x,N})$ with $J \equiv \hbar I/2e$. As noted in [3], equation (2) is the Hamiltonian for arrays of the *washboard potential*, commonly used for a single junction [8], and describes non-equilibrium behaviour of the junction arrays driven by external currents. The corresponding action can be written in the form

$$\begin{aligned} A[\phi] &= \sum_{\langle ij\rangle} [K_0 \cos(\phi_i - \phi_j) + K_{ij}(\phi_i - \phi_j)] \\ &\equiv \sum_{\langle ij\rangle} V_{ij}(\phi_i - \phi_j) \end{aligned} \tag{3}$$

with $K_0 \equiv J_0/k_\mathrm{B}T \equiv \beta J_0$ and $K_{ij} \equiv \beta J \delta_{j_x, i_x+1} \equiv K\delta_{j_x, i_x+1}$. The phase transition associated with the action (3) has been studied via the renormalization group technique, and concluded to be a KT transition at the critical temperature renormalized by the external currents [5].

We then calculate the $I-V$ characteristics with the action in equation (3). The potential difference across the array (between edges $i_x=0$ and $i_x=N$) is given by

$$V=\frac{R}{N}\sum_{i_y}\left(\left\langle\frac{\mathrm{d}\phi_{(0,i_y)}}{\mathrm{d}t}\right\rangle_t-\left\langle\frac{\mathrm{d}\phi_{(N,i_y)}}{\mathrm{d}t}\right\rangle_t\right)$$
$$=\frac{R}{N}\sum_{i_y}(\langle h_{(0,i_y)}\rangle_t-\langle h_{(N,i_y)}\rangle_t) \tag{4}$$

where t is again the (dimensionless) rescaled time, and $\langle\ldots\rangle_t$ represents the time average during the measurement time. In the stationary state, we replace the time average by the thermal average with respect to the action $A[\phi]$, and write

$$\langle h_i\rangle_t=\sum_j G_{ij}I_j-\sum_j\sum_k{}' G_{ij}\langle\sin(\phi_j-\phi_k)\rangle \tag{5}$$

where

$$\langle O\rangle\equiv\frac{1}{Z}\int\left(\prod_k \mathrm{d}\phi_k\right)O\exp(A[\phi])$$

with the partition function $Z\equiv\int(\Pi_k\,\mathrm{d}\phi_k)\exp(A[\phi])$. Thus the voltage is related to the spin-spin correlation function $\Gamma_{jk}\equiv\langle\exp\mathrm{i}(\phi_j-\phi_k)\rangle$ between nearest neighbours j and k, whose imaginary part gives the second term in equation (5).

To calculate the correlation function, we apply the dual transformation to

$$A_s[\phi]\equiv\sum_{\langle lm\rangle}[K_0\cos(\phi_l-\phi_m)+(K_{lm}+\mathrm{i}\delta_{li}\delta_{mj})(\phi_l-\phi_m)]$$
$$=\sum_{\langle lm\rangle}V_{lm}(\phi_l-\phi_m)+\mathrm{i}(\phi_i-\phi_j). \tag{6}$$

Following [5], we regard $V_{lm}(\theta)$ as a periodic function with period $2n\pi$ (with the limit $n\to\infty$ to be taken), and write $\exp(V_{lm}(\theta)+\mathrm{i}\theta\delta_{li}\delta_{mj})$ as a Fourier series: $\exp(V_{lm}(\theta)+\mathrm{i}\theta\delta_{li}\delta_{mj})=\Sigma_s\exp[\mathrm{i}s\theta/n+\tilde{V}_{lm}(s/n-\delta_{li}\delta_{mj})]$, where, for those bonds with $K_{ij}=0$, s takes only values of multiples of n. The correlation function thus can be written in the form

$$\Gamma_{ij}=\frac{1}{Z}\sum_{m_l}\left(\prod_l\int_{-\infty}^{\infty}\mathrm{d}\phi_l\right)\exp\left[\sum_{\langle lm\rangle}\tilde{V}_{lm}(\phi_l-\phi_m-\delta_{li}\delta_{mj})+\sum_l 2\pi\mathrm{i}\,m_l\phi_l\right]. \tag{7}$$

In the Villain approximation, which is valid at low temperatures and for small currents [5], $\tilde{V}_{ij}(s)$ takes the form

$$\exp\{\tilde{V}_{ij}(s)\}=\frac{1}{(2\pi\tilde{K}_0)^{1/2}}\exp\left[-\frac{1}{2\tilde{K}_0}s^2+\mathrm{i}\frac{\tilde{K}_{ij}}{\tilde{K}_0}s\right] \tag{8}$$

leading to the spin-spin correlation function

$$\Gamma_{ij}=\frac{1}{Z}\sum_{m_l}\left(\prod_l\int_{-\infty}^{\infty}\mathrm{d}\phi_l\right)\exp\left[-K_0\sum_{lm}(\cos\theta_0\,\delta_{m+\hat{x},l}+\delta_{m+\hat{y},l})(\phi_l-\phi_m-\delta_{li}\delta_{mj})^2\right.$$
$$\left.+2\pi\mathrm{i}K_0\sqrt{\cos\theta_0}\sum_l m_l\phi_l-\mathrm{i}\frac{\tilde{K}_{ij}}{\tilde{K}_0}\right]$$
$$\equiv\exp\left(-\mathrm{i}\frac{\tilde{K}_{ij}}{\tilde{K}_0}\right)\Gamma_0 \tag{9}$$

with $\tilde{K}_0\equiv K_0\cos\theta_0$ and $\tilde{K}_{ij}\equiv K_{ij}\theta_0\cos\theta_0$. In equation (9), $\Gamma_0\equiv\langle\mathrm{e}^{\mathrm{i}(\phi_i-\phi_j)}\rangle_{I=0}$ is the spin-spin correlation function between nearest neighbours i and j, in the absence of the external current. It is real and independent of the site indices i and j by symmetry.

In this low-current regime, equation (4) together with equations (5) and (9) then leads to the potential difference across the array

$$V \approx R(1-\Gamma_0)(NI) \tag{10}$$

which displays the linear Ohmic behaviour unless Γ_0 is equal to unity. The isotropic correlation function Γ_0 is given by the standard KT theory [4]:

$$\Gamma_0 = \begin{cases} \exp(-1/(4K_r)) & \text{for } T < T_c \\ 0 & \text{for } T > T_c \end{cases} \tag{11}$$

with the renormalized interaction

$$K_r = \left\{ \frac{1}{K_0\sqrt{\cos\theta_0}} - \frac{\pi^3 \exp(-\pi^2 K_0 \sqrt{\cos\theta_0})}{1-\pi K_0\sqrt{\cos\theta_0}/2} \right\}^{-1}$$

which shows $\Gamma_0 \neq 1$ except at zero temperature. Thus, it is concluded that the I–V relation is linear for low currents at all finite temperatures.

To confirm this result, we have performed numerical simulations with the Langevin equations (1) at various temperatures and applied currents, computing the average voltage drop at each run. In equation (1), the lattice Green function G is singular due to the existence of the U(1) symmetry of the system: a uniform rotation of all the phases leaves the equation of motion invariant. To remedy this problem, we fix the phase of one superconducting grain as in [3], which amounts to grounding that grain.

The equations of motion (1) are integrated with discrete time steps of $\Delta t = 0.1$. At each run, $N_t = 10^5$–10^6 time steps per spin were used to compute averages. Both Δt and N_t were varied to verify that the steady state was achieved. For example, discrete time steps of $\Delta t = 0.01$ were used at various temperatures and currents, but no appreciable change could be observed in the numerical data. Standard block averaging and independent runs were used for estimating sampling errors. All calculations were performed with vectorized programs on a Cray.

The resulting I–V curves of the array of size $N = 16$ at various temperatures are shown in figure 1 while figure 2 displays typical dependence of the corresponding exponent α in the I–V characteristics ($V \propto I^{\alpha}$). These figures show that the I–V relation is linear at high temperatures (regardless of currents) or at high currents ($I > I_c$) (regardless of the temperature), which is expected since at high temperatures

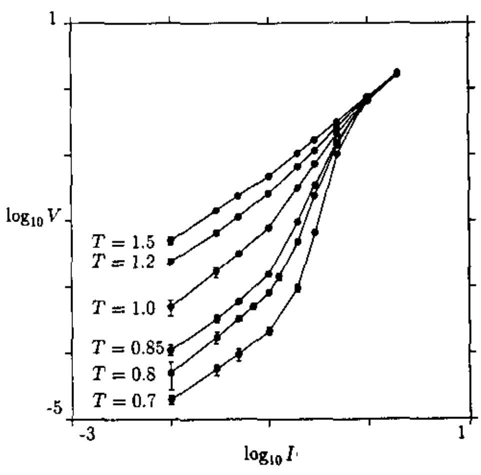

Figure 1. I–V characteristics of the $N = 16$ array at various temperatures T. The voltage drop V and the external current I are scaled in units of NI_cR and I_c, respectively. Lines are merely guides to the eye.

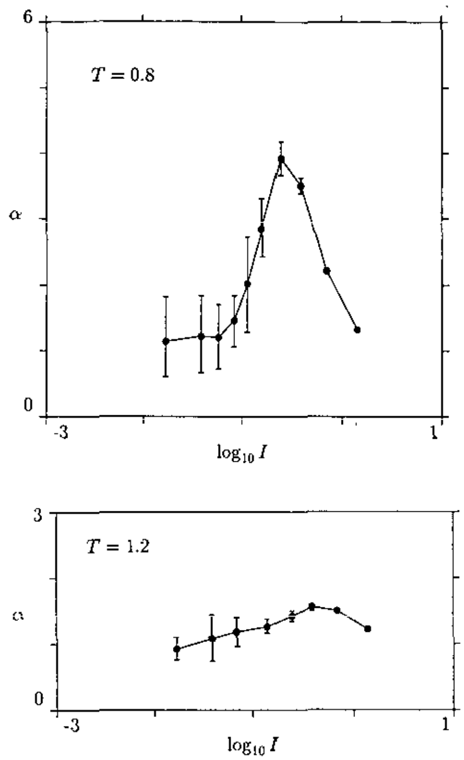

Figure 2. Dependence of exponent α in the $I-V$ characteristics ($V \propto I^{\alpha}$) on the applied current I, at various temperatures T. I is again scaled in units of I_c, and lines are merely guides to the eye.

or at high currents the array should be in the normal state [5]. As the currents are decreased below I_c, the $I-V$ relation becomes nonlinear at low temperatures, in agreement with experiments [6] and previous simulations [3]. The exponent α grows from unity as the temperature is lowered. In particular, the peak value of α apparently becomes 3 (and gets larger) as the temperature approaches around 0.9 (and decreases below this value), which is the estimated critical temperature T_c in the equilibrium analysis [5]. At first sight, this change of α from unity to 3 at T_c seems to agree with the dynamical KT theory [7]. However, it should be noted that this nonlinear $I-V$ relation does not persist as the currents are decreased further. Remarkably, at low currents ($I < 0.05 I_c$) the $I-V$ relation becomes linear again at all finite temperatures, thus confirming the analytical result given by equation (10).

In order to check size dependence, we have also studied arrays of size $N = 8$, 12, 24 and 32 in addition to $N = 16$. Figure 3 shows the resulting $I-V$ curves at $T = 0.7$, all but the smallest ($N = 8$) of which appear to display essentially the same behaviour. Thus the $N = 16$ array is considered to exhibit bulk properties reasonably well, without excessive finite-size effects. To see the detail of finite-size effects, we have calculated the corresponding array resistance $\tilde{R} \equiv \mathrm{d}V/N\,\mathrm{d}I$, which, in the case of the nonlinear $I-V$ relation, should vanish in the low-current limit ($I \to 0$). In figure 4, which shows the size dependence of $\tilde{R}$, it seems rather unlikely that $\tilde{R}$ approaches zero or $\log_{10} \tilde{R}$ approaches minus infinity as N gets larger. Instead $\tilde{R}$ appears to approach a finite value as $N \to \infty$, suggesting that the linear $I-V$ relation is not a mere finite-size effect.

This not only provides a natural explanation of the anomalous linear $I-V$ characteristics at low currents, observed both in junction arrays [6] and in thin films [9], but also suggests a slightly modified criterion of the phase transition in the $I-V$

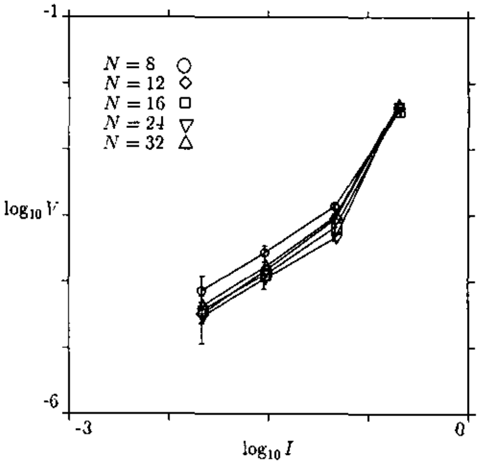

Figure 3. $I-V$ characteristics of arrays of various size at temperature $T=0.7$. Lines are merely guides to the eye.

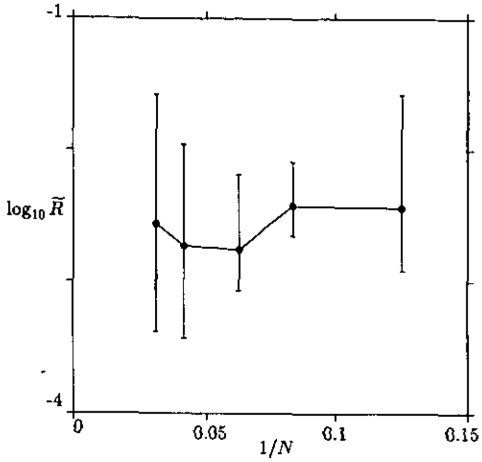

Figure 4. Dependence of the array resistance $\tilde{R}$ on the size N. $\tilde{R}$ is scaled in units of the shunt resistance R, and the line is again merely a guide to the eye.

characteristics. In the conventional dynamic KT theory, which is based on the superfluid analogy with intrinsic phenomenological characters [7], the phase transition is characterized by the jump in the exponent α (from unity to 3) in the low-current limit ($I \to 0$). In contrast, this work shows that the nonlinear $I-V$ relation only appears at intermediate currents and accordingly that *the change of α, again characterizing the phase transition, should be investigated in the intermediate current regime where α reaches its maximum.* Physically, this implies that finite currents are necessary for unbinding vortex–antivortex pairs and thus producing nonlinear behaviour, presumably due to the existence of finite potential barriers.

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